J. Fluid Mech. (1966), *vol.* 24, *part* 3, *pp.* 577–595
Printed in Great Britain

The steady flow of a viscous fluid past a flat plate

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(Received 1 September 1964 and in revised form 21 July 1965)

Numerical methods are used to investigate the steady two-dimensional motion of a viscous incompressible fluid past a flat plate of finite breadth at zero incidence to a uniform stream. Before application of numerical techniques, the governing partial differential equations for the stream function and vorticity are reduced to ordinary differential equations by an adaptation of methods normally used to solve Oseen's linearized equations. The complete range of the Reynolds number R is considered, from indefinitely small to indefinitely large. All the results are intended to represent solutions of the full Navier–Stokes equations of motion, although in practice approximations are inevitable. These are mainly brought about by the necessity of limiting the size of the calculations.

At the lower end of the Reynolds-number range, the calculated frictional drag coefficient agrees well with the results of Tomotika & Aoi (1953) based on Oseen's equations. At intermediate and higher Reynolds numbers there is good agreement with the experimental results of Janour (1951) and with the improvement of the Blasius solution given by Kuo (1953). Finally a limiting solution is obtained as $R \to \infty$. This shows that the drag coefficient is proportional to $R^{-\frac{1}{2}}$, in accordance with boundary-layer theory. The actual calculated value of the coefficient is about 4 % higher than the Blasius value.

Although the present results tend generally to confirm the trend of the recently published results at $R = 0{\cdot}1$, 1 and 10 of Janssen (1957), there are substantial discrepancies in the detailed results in a number of instances. In particular, the drag values obtained at $R = 1$ and 10 are some 20 % higher than Janssen's although there is reasonable agreement at $R = 0{\cdot}1$. It seems possible that Janssen's analogue is a little crude at the higher Reynolds numbers.

1. Introduction and basic equations

The Navier–Stokes equations describing the steady motion of a viscous, incompressible, fluid are, in terms of the pressure p, the density ρ and the velocity vector $\mathbf{q}$,

$$(\mathbf{q}\,.\,\nabla)\,\mathbf{q} = -\rho^{-1}\,\mathrm{grad}\,p + \nu\nabla^2\mathbf{q}, \tag{1}$$

where ν is the coefficient of kinematical viscosity. The equation of continuity is

$$\mathrm{div}\,\mathbf{q} = 0, \tag{2}$$

and if the motion is two-dimensional with Cartesian velocity components (u, v), equation (2) may be satisfied by introducing a stream function $\psi(x, y)$ such that

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x. \tag{3}$$

By eliminating the pressure from (1), we obtain the single equation

$$\nu\nabla_1^2\zeta = \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} \tag{4}$$

for the scalar vorticity

$$\zeta = \partial v/\partial x - \partial u/\partial y, \tag{5}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Moreover, equations (3) and (5) together give

$$\nabla_1^2\psi + \zeta = 0. \tag{6}$$

Equations (4) and (6) are the simultaneous equations which are usually solved numerically when the steady motion past an obstacle in a uniform stream is required. If the stream has velocity components $(U, 0)$ the boundary conditions are

$$\partial\psi/\partial y \to U, \quad \partial\psi/\partial x \to 0, \tag{7}$$

at large distances from the obstacle, together with the no-slip condition $\mathbf{q} = 0$ on the body itself. Numerical solutions have so far been mainly confined to motion past circular cylinders, e.g. Thom (1933), Kawaguti (1953), Allen & Southwell (1955) and Apelt (1961). For a flat plate of finite breadth at zero incidence to the stream, a few results, at Reynolds numbers 0·1, 1 and 10, have been given by Janssen (1957) using an electrical analogue of the numerical problem.

The flat plate is a suitable problem in which to investigate the complete Reynolds number range numerically, since limiting solutions are known both for low and high Reynolds numbers. On the one hand there is the linearized theory of Oseen. Tomotika & Aoi (1953) have given a reliable solution of Oseen's equations in this case. At the other end of the scale there is the Blasius theory, although strictly this applies to a plate of infinite breadth. Kuo (1953) has modified the Blasius formula for the resistance to apply to a finite plate at moderate Reynolds numbers. No independent check on this result has yet been given.

In the intermediate range between the extremes of very low and moderately high Reynolds numbers the only theoretical results available are Janssen's. His results for the drag coefficient seem to be rather low compared with, on the one hand, Tomotika & Aoi's solution at $R = 1$ and on the other, the experimental results in the neighbourhood of $R = 10$ due to Janour (1951). Janour's measurements extend over the range $R = 12$ to 2335. At the lower end they are therefore largely unconfirmed by theory. At the upper end they clearly approach the value predicted by the Blasius theory. Although Janssen suggests that his own results confirm the trend towards the Blasius solution as R is increased, this is perhaps a little optimistic at so low a Reynolds number as 10 and there is hardly any tendency to agree with Kuo's modification at this value of R. There is clearly scope for further investigation of the theoretical problem at all Reynolds numbers. Such an investigation is presented in this paper. Numerical solutions of equations (4) and (6) are obtained by applying a technique which is similar to the method of separation of variables, and which is suggested by the analytical treatment of Oseen's linearized equations. As in the case of Oseen's equations, a prior step in the analysis is the transformation of the co-ordinate system to one suitable to the obstacle concerned.

For a flat plate of length $2c$ situated along the x-axis with edges at $x = \pm c$, the transformation is

$$x = c\cosh\xi\cos\eta, \quad y = c\sinh\xi\sin\eta. \tag{8}$$

The upper half of the (x, y)-plane, which by symmetry is all that need be considered, is transformed to the semi-infinite strip $\xi \geqslant 0$, $0 \leqslant \eta \leqslant \pi$. The plate transforms to $\xi = 0$, with trailing edge at $\eta = 0$ and leading edge at $\eta = \pi$. The transformation of derivatives is

$$\left.\begin{aligned} \partial/\partial\xi &= c(\sinh\xi\cos\eta\,\partial/\partial x + \cosh\xi\sin\eta\,\partial/\partial y), \\ \partial/\partial\eta &= c(-\cosh\xi\sin\eta\,\partial/\partial x + \sinh\xi\cos\eta\,\partial/\partial y). \end{aligned}\right\} \tag{9}$$

Transforming equations (4) and (6) and introducing the dimensionless stream function and vorticity and the Reynolds number defined by the equations

$$\psi = Uc\psi', \quad \zeta = U\zeta'/c, \quad R = 2Uc/\nu, \tag{10}$$

we obtain, after suppressing primes, the equations

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2} = \frac{R}{2}\left(\frac{\partial\psi}{\partial\eta}\frac{\partial\zeta}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\eta}\right), \tag{11}$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} + \tfrac{1}{2}(\cosh 2\xi - \cos 2\eta)\,\zeta = 0. \tag{12}$$

Boundary conditions are readily deduced from (9). Since $u = v = 0$ on the plate

$$\psi = \partial\psi/\partial\xi = 0, \quad \text{when} \quad \xi = 0. \tag{13a}$$

At large distances, from (7),

$$\frac{1}{\cosh\xi\sin\eta}\frac{\partial\psi}{\partial\xi} \to 1, \quad \frac{1}{\sinh\xi\cos\eta}\frac{\partial\psi}{\partial\eta} \to 1, \quad \text{as} \quad \xi \to \infty. \tag{13b}$$

Finally by symmetry $\quad \psi = \zeta = 0, \quad \text{when}\ \eta = 0, \pi. \qquad (13c)$

In these relations, and subsequently throughout the paper, ψ and ζ are dimensionless; other quantities will be supposed dimensional. For convenience we shall write $\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 = \nabla^2$, although this is not the true Laplacian in the (ξ, η)-plane.

2. Flow at large distances and method of analysis

A method of analysis is suggested by the Oseen linearized treatment of the problem. The following is true for any body symmetrical about the x-axis, since for the moment only the flow outside a large elliptic contour $\xi = \xi_0$ is considered independently of the inner boundary conditions (13a). From (13b), as $\xi \to \infty$

$$\partial\psi/\partial\xi \sim \tfrac{1}{2}e^{\xi}\sin\eta, \quad \partial\psi/\partial\eta \sim \tfrac{1}{2}e^{\xi}\cos\eta,$$

and the Oseen linearized form of (11) is

$$\nabla^2\zeta = \tfrac{1}{4}R\, e^{\xi}(\cos\eta\,\partial\zeta/\partial\xi - \sin\eta\,\partial\zeta/\partial\eta). \tag{14}$$

Substituting
$$\zeta = \phi\, e^{F(\xi,\eta)} \tag{15}$$

with
$$F(\xi,\eta) = \tfrac{1}{8}R\, e^{\xi}\cos\eta, \tag{16}$$

then ϕ must satisfy
$$(\nabla^2 - \tfrac{1}{64}R^2 e^{2\xi})\,\phi = 0. \tag{17}$$

Fundamental solutions of (17) are

$$K_n(\beta)\sin n\eta, \quad K_n(\beta)\cos n\eta, \quad I_n(\beta)\sin n\eta, \quad I_n(\beta)\cos n\eta,$$

with
$$\beta(\xi) = \tfrac{1}{8}R\, e^{\xi},$$

where I_n and K_n are modified Bessel functions of the first and second kinds respectively. By virtue of the properties of these functions for large β, only the first two solutions are admissible if ζ is to remain finite as $\xi \to \infty$ and of these only the first satisfies (13c). Summing over integer values of n, a complete solution in the range $\eta = 0$ to $\eta = \pi$ is given by

$$\phi(\xi,\eta) = \sum_{n=1}^{\infty} g_n(\xi)\sin n\eta, \tag{18}$$

with
$$g_n = A_n K_n(\beta), \tag{19}$$

where the A_n are arbitrary constants.

A similar form of solution of (12), complete in the range $\eta = 0$ to $\eta = \pi$ and satisfying (13c), may be taken as

$$\psi(\xi,\eta) = \sum_{n=1}^{\infty} f_n(\xi)\sin n\eta. \tag{20}$$

We then have
$$f_n(\xi) = \frac{2}{\pi}\int_0^{\pi} \psi(\xi,\eta)\sin n\eta\, d\eta,$$

i.e. apart from a factor, $f_n(\xi)$ is the Fourier sine transform of ψ. Integrating twice by parts with regard to η and using the conditions $\psi(\xi,0) = \psi(\xi,\pi) = 0$, we find that the transform of (12) is

$$f_n'' - n^2 f_n + r_n(\xi) = 0 \quad (n = 1, 2, 3, \ldots), \tag{21}$$

where
$$r_n(\xi) = \frac{1}{\pi}\int_0^{\pi} (\cosh 2\xi - \cos 2\eta)\,\zeta\sin n\eta\, d\eta, \tag{22}$$

and primes denote differentiation with regard to ξ. Substituting in this integral for ζ using (15) and in turn for ϕ from (18) and letting

$$\mathscr{I}_n(\xi) = \frac{1}{\pi}\int_0^{\pi} e^{F(\xi,\eta)}\cos n\eta\, d\eta, \tag{23}$$

we obtain

$$r_n(\xi) = \frac{1}{2}\sum_{p=1}^{\infty} g_p\{(\mathscr{I}_{n-p} - \mathscr{I}_{n+p})\cosh 2\xi - \tfrac{1}{2}(\mathscr{I}_{n-p-2} + \mathscr{I}_{n-p+2} - \mathscr{I}_{n+p-2} - \mathscr{I}_{n+p+2})\}. \tag{24}$$

Clearly $\mathscr{I}_{-n} = \mathscr{I}_n$ for all n.

Boundary conditions for the f_n as $\xi \to \infty$ follow from (13*b*). Substituting for ψ from (20), necessary and sufficient conditions for satisfaction of (13*b*) are

$$e^{-\xi} f_1(\xi) \to \tfrac{1}{2}, \quad e^{-\xi} f_n(\xi) \to 0 \quad (n \neq 1). \tag{25}$$

The present object is to deduce a form of the solutions for the $f_n(\xi)$ as $\xi \to \infty$, but it may be observed that (21) and (24) are valid for all ξ provided $F(\xi, \eta)$ and the functions $g_n(\xi)$ are suitably chosen to ensure the satisfaction of the full equation (11) rather than the linearized form, a step shortly to be considered.

A first integral of (21) is

$$f_n' + nf_n + C_n e^{n\xi} = e^{n\xi} \int_\xi^\infty e^{-nt} r_n(t)\, dt, \tag{26}$$

provided the integral on the right-hand side exists. This may be seen by considering the form of $r_n(\xi)$ as $\xi \to \infty$. With $F(\xi, \eta)$ given by (16), equation (23) defines exactly the modified Bessel function of the first kind with argument β, viz.

$$\mathscr{I}_n(\xi) \equiv I_n(\beta),$$

and the functions $g_n(\xi)$ are given, for large enough ξ, by (19). Using the asymptotic expansions of the modified Bessel functions for large β it may readily be shown from (24) that, as $\xi \to \infty$,

$$r_n(\xi) \to \frac{16n}{R^2} \sum_1^\infty pA_p = nC,$$

where C is constant, assuming the series to converge. Subject to this assumption, the integral in (26) exists and if we divide each side of (26) by $e^{n\xi}$ and let $\xi \to \infty$, we find using (25) that

$$C_1 = -1, \quad C_n = 0 \quad (n = 2, 3, 4, \ldots). \tag{27}$$

It then follows that, as $\xi \to \infty$,

$$f_1(\xi) \sim \tfrac{1}{2} e^{\xi} + C, \quad f_n(\xi) \to C/n \quad (n \neq 1).$$

Substituting in (20) and summing the series, it is found that except at $\eta = 0$, at which a finite discontinuity exists,

$$\psi(\xi, \eta) \sim \tfrac{1}{2} e^{\xi} \sin \eta + K(1 - \eta/\pi), \tag{28}$$

where $K = \frac{1}{2}\pi C$.

This result for the limiting form of ψ is in accordance with the general result of Imai (1951) obtained by different methods. The boundary conditions assumed by Janssen in his treatment of the flat-plate problem do not accord with this. He combines (13*b*) in the form

$$\psi = \sinh \xi_\infty \sin \eta, \quad \xi = \xi_\infty,$$

and enforces this exactly at a chosen ξ_∞ which, in practice is, of course, finite. Allen & Southwell, in dealing with the flow past a circular cylinder, state the condition for the (dimensional) stream function as $\psi \to Uy$ at large distances. Kawaguti, however, uses the relationship that, as $r \to \infty$,

$$\psi(r, \theta) \sim r \sin \theta - \tfrac{1}{2} C_D (1 - \theta/\pi), \tag{29}$$

where (r, θ) are polar co-ordinates and C_D is the drag coefficient. This is in effect equivalent to (28), as may be shown by calculating a value for K.

If D is the drag on the body, i.e. the force in the direction of the undisturbed stream, and D_1 is the force in the same direction exerted by the fluid outside a closed contour C' surrounding the body on the fluid inside it, the momentum equation in this direction for the enclosed fluid is

$$D_1 - D = \int_{C'} \rho u(u\,dy - v\,dx). \tag{30}$$

In customary notation for the stress components in the fluid

$$\begin{aligned} D_1 &= \int_{C'} (lp_{xx} + mp_{xy})\,ds \\ &= -\int_{C'} \left\{ l\left(p - 2\mu\frac{\partial u}{\partial x}\right) - \mu m\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right\} ds, \end{aligned}$$

where (l, m) are the direction cosines of the outward-drawn normal to C', s is arc length measured in the anti-clockwise sense and μ is the viscosity. In particular if C' coincides with the contour C'' of the body itself then

$$D = D_1 = -\int_{C''} \left\{ lp + \mu m\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\} ds. \tag{31}$$

The two parts of this integral, depending respectively on the pressure and the viscosity, give the pressure drag and frictional drag.

The total outflow across the contour C' must be zero. Goldstein (1929) has shown that for a large contour (or closed surface surrounding the body in the case of a closed body) equation (30) reduces to

$$D = \rho U I,$$

where I is the outflow over the large contour, balanced by an equal inflow along the wake. At large distances downstream it follows from Oseen theory (viz. by considering the form of ζ given by (15) for large ξ) that the wake coincides with $\eta = 0$. Hence

$$I = Uc\int_{C'} \left(\frac{\partial \psi}{\partial s}\right)_{\xi\to\infty} ds = Uc[\psi(\infty, \eta)]_0^{2\pi},$$

in terms of the dimensionless ψ. Substituting from (28)

$$I = -2KUc,$$

i.e.

$$K = -D/2\rho U^2 c = -C_D,$$

where C_D is the drag coefficient. The difference in the factor in Kawaguti's result (29) is due to the slight difference in definition of C_D for a circular cylinder. The one used here is customary for a flat plate.

It follows from these results that if the many-valued function of η is not present in (28), integration round a large contour must yield zero drag. This must be true of Janssen's contour $\xi = \xi_\infty$, although what influence this may have on the numerical solution is not easy to see.

It now remains to modify the definitions of the functions $F(\xi, \eta)$ and $g_n(\xi)$ so that the full equation (11) shall be satisfied throughout the complete range $\xi = 0$ to $\xi = \infty$, at the same time preserving the details of the Oseen solution for large ξ.

Essentially, this requires that the substitution (15) for ζ shall be made in (11) rather than (14). As already noted, the set of equations (21) can then be made to represent (12) over the complete range of ξ. Boundary conditions for $f_n(\xi)$ at $\xi = 0$ are deduced from (13*a*). They are that

$$f_n(0) = f'_n(0) = 0 \quad (n = 1, 2, 3, \ldots). \tag{32}$$

The function $F(\xi, \eta)$ is chosen largely for convenience, but subject to the important restriction that it should reduce to (16) for large ξ. This ensures that $g_n(\xi)$ reduce to (19) and the behaviour of the functions for large ξ is known. A particularly convenient definition of $F(\xi, \eta)$ is to make it a solution of

$$\partial F/\partial \xi = \tfrac{1}{4} R\, \partial \psi / \partial \eta, \tag{33}$$

whence from (13*b*) as $\xi \to \infty$

$$F(\xi, \eta) \sim \tfrac{1}{8} R\, e^{\xi} \cos \eta$$

as required. Furthermore, if with this choice of F we substitute (15) into (11), the latter becomes

$$\nabla^2 \phi + P(\xi, \eta)\, \partial \phi / \partial \eta + Q(\xi, \eta)\, \phi = 0, \tag{34}$$

with

$$P(\xi, \eta) = 2 \frac{\partial F}{\partial \eta} + \tfrac{1}{2} R \frac{\partial \psi}{\partial \xi}, \tag{35}$$

$$Q(\xi, \eta) = \nabla^2 F + \frac{\partial F}{\partial \eta} \left(\frac{\partial F}{\partial \eta} + \tfrac{1}{2} R \frac{\partial \psi}{\partial \xi} \right) - \frac{R^2}{16} \left(\frac{\partial \psi}{\partial \eta} \right)^2. \tag{36}$$

It will subsequently appear that the important point about (34) is that there is no term in $\partial \phi / \partial \xi$.

Supposing for the moment that P and Q are specified functions of ξ and η, equation (34) is linear in ϕ. We cannot separate the variables as in (17) but in effect $g_n(\xi)$, equation (18), is the Fourier sine transform of $\phi(\xi, \eta)$, viz.

$$g_n(\xi) = \frac{2}{\pi} \int_0^{\pi} \phi \sin n\eta \, d\eta.$$

Integrating twice by parts with regard to η and using the conditions

$$\phi(\xi, 0) = \phi(\xi, \pi) = 0,$$

we see that the transform of (34) is

$$\tfrac{1}{2}\pi (g''_n - n^2 g_n) + \int_0^{\pi} \left(P \frac{\partial \phi}{\partial \eta} + Q \phi \right) \sin n\eta \, d\eta = 0, \tag{37}$$

holding for $n = 1, 2, 3, \ldots$, where again primes denote differentiation with regard to ξ. In order to proceed further, series must be obtained for P and Q in terms of the functions $f_n(\xi)$ and their derived functions and substituted in the integral in (37), which is then evaluated using term-by-term integration. The set of equations (37) is then reduced to the form

$$g''_n - n^2 g_n + \sum_{p=1}^{\infty} k_{n,p}\, g_p = 0, \tag{38}$$

where the k's are functions of ξ to be determined. Since $\partial \phi / \partial \xi$ is absent from (37) there are no terms involving $g'_n(\xi)$ in (38), which makes them especially suitable for solution by numerical methods. Further, their solutions must approach (19) as $\xi \to \infty$, provided the outer boundary condition for ψ is satisfied properly.

Conditions for the validity of term-by-term differentiation of the Fourier series, together with the method of obtaining derived series, are given by Jeffreys & Jeffreys (1962). It follows from these that

$$\frac{\partial \psi}{\partial \eta} = \sum_{n=1}^{\infty} n f_n(\xi) \cos n\eta. \tag{39}$$

A series for $F(\xi, \eta)$ may now be obtained by substituting this result into (33) and integrating term by term with regard to ξ. This may be taken as

$$F(\xi, \eta) = \tfrac{1}{4} R \sum_{n=1}^{\infty} \beta_n(\xi) \cos n\eta, \tag{40}$$

with
$$\beta_n(\xi) = n \int_0^{\xi} f_n(t)\, dt, \tag{41}$$

i.e. we choose that particular solution of (33) that satisfies the condition $F(0, \eta) = 0$.

Further derived series necessary for the term-by-term evaluation of the integral in (37) can now be obtained. It is hardly necessary to give them in detail or describe any of the further steps in the process. It may be verified that a formula for the coefficients $k_{n,p}(\xi)$ can conveniently be given in terms of four sets of coefficients defined respectively by

$$\left.\begin{aligned} a_n(\xi) &= n\beta_n, \quad b_n(\xi) = 2f_n' - n\beta_n, \\ c_n(\xi) &= f_n' - n\beta_n, \quad d_n(\xi) = nf_n \end{aligned}\right\} \quad (n = 1, 2, 3, \ldots). \tag{42}$$

The formula is

$$k_{n,p}(\xi) = \tfrac{1}{8} R\{(p-n)\, c_{n+p} + (n+p)\, c_{n-p}\} - \tfrac{1}{64} R^2 \sum_{r=1}^{\infty} (M_r + N_r), \tag{43}$$

where
$$M_r = d_r(d_{p+r-n} + d_{n+r-p} - d_{n+p-r} - d_{n+p+r}) \tag{44}$$

and
$$N_r = a_r(b_{p+r-n} + b_{n+r-p} + b_{n+p-r} - b_{n+p+r}). \tag{45}$$

Since some of the coefficients in these expressions occur with negative or zero suffixes, the definitions (42) are extended to take account of this according to the rules
$$b_{-n} = -b_n, \quad c_{-n} = -c_n, \quad d_{-n} = -d_n, \quad (n = 1, 2, 3, \ldots),$$
with
$$b_0 = c_0 = d_0 = 0.$$

Boundary conditions for the equations (38) must now be stated. These may be considered in the following way. If the equation (17) were valid from $\xi = 0$ to $\xi = \infty$, the solution for the functions $g_n(\xi)$ would be the Oseen solution given by (19). Here each function has an associated arbitrary constant. This expresses the fact that the functions $g_n(\xi)$ are independent solutions of the equations

$$g_n'' - (n^2 + \tfrac{1}{64} R^2 e^{2\xi})\, g_n = 0, \tag{46}$$

which is the special form that (38) would take in this case. A condition to determine these constants would then be found by putting $\xi = 0$ in (26). Using (27) and (32) this gives

$$\left.\begin{aligned} \int_0^\infty e^{-n\xi} r_n(\xi)\, d\xi &= -1 \quad (n=1) \\ &= 0 \quad (n=2,3,\ldots). \end{aligned}\right\} \tag{47}$$

These conditions may be recognized as equivalent to the conditions which are used to find a similar set of constants in the various applications of Oseen linearized theory, cf. Tomotika & Aoi (1950, 1953), although the process is not exactly the same.

In the present case the governing equation is (34) and the functions $g_n(\xi)$ satisfy the generalized set of equations (38). However, in the following section a set of independent numerical solutions, each with an associated arbitrary constant, will be considered. These are obtained by treating (38), at a given stage of an iterative numerical process, as a set of linear equations for the functions $g_n(\xi)$ with numerical coefficients $k_{n,p}(\xi)$. With this extension of the meaning of the functions $g_n(\xi)$, the same condition (47) is valid to determine these constants.

Next, the condition as $\xi \to \infty$ must be considered. In the Oseen case there is no problem, since the functions (19) automatically satisfy the correct condition that the scalar vorticity ζ tends to zero for large ξ. In the present case, provided the condition (47) and the initial conditions (32) have been satisfied, the functions $f_n(\xi)$ which satisfy (21) automatically tend to the form consistent with (28) and, consequently, the equations (38) approach a form similar to (46). This shows itself in that $k_{n,n}(\xi) \sim -(\frac{1}{64}R^2)\, e^{2\xi}$, while for $n \neq p$, $k_{n,p}(\xi)/k_{n,n}(\xi) \to 0$, i.e. the diagonal terms of the matrix of coefficients $k_{n,p}(\xi)$ dominate (38) for large ξ. In the method of numerical solution the problem of the condition for the $g_n(\xi)$ as $\xi \to \infty$ is therefore solved in the following way. For large enough ξ, equations (38) may adequately be approximated by

$$g_n'' - \chi(\xi)\, g_n = 0, \tag{48}$$

with

$$\chi(\xi) = n^2 - k_{n,n}(\xi).$$

However, χ itself becomes very large if ξ is large, and this makes the numerical finite-difference approximations which are used inaccurate, so a further approximation to (48) is made by employing the method of Jeffreys & Jeffreys (1962) to obtain the approximate solution

$$g_n \approx D_n \chi^{-\frac{1}{4}} \exp\left\{-\int_{\xi_1}^{\xi} \chi^{\frac{1}{2}}\, d\xi\right\}, \tag{49}$$

where ξ_1 is a sufficiently large value of ξ. This is used for the solution for $\xi > \xi_1$ and the arbitrary constants D_n are chosen to satisfy conditions imposed at $\xi = \xi_1$ in a manner which will be described. By virtue of the form assumed by $k_{n,n}(\xi)$ for large ξ, it is known that (49) approximates more accurately to (48) as ξ increases.

Once numerical solutions have been obtained, the drag on the plate may be found by evaluating (31) round the plate itself. The pressure drag is zero in this case and the drag coefficient becomes

$$C_D = \frac{D}{2\rho U^2 c} = -\frac{2}{R}\int_0^\pi \zeta_0 \sin\eta\, d\eta,$$

the subscript referring to $\xi = 0$. Substituting in the integral using (15) and (18) and noting that $F(0, \eta) = 0$, we find

$$C_D = -(\pi/R)\, g_1(0). \tag{50}$$

The local shearing stress at the plate is $p_{xy} = \mu(\partial u/\partial y)_0$. Defining the local coefficient of skin friction c_f to be the dimensionless shearing stress $p_{xy}/\rho U^2$, then

$$c_f = -2\zeta_0/R. \tag{51}$$

3. Method of numerical solution

The numerical process consists of an iterative cycle of three stages which, starting from an initial assumption, is repeated successively until convergence is obtained.

In the first stage, assuming the $r_n(\xi)$ are known, equations (21) are solved subject to (32) by step-by-step integration. From the computed solutions for the $f_n(\xi)$, the coefficients $k_{n,p}(\xi)$ are calculated, using numerical differentiation and integration to obtain $f_n'(\xi)$ and $\beta_n(\xi)$. The second stage of the process now consists of finding numerical solutions of (38) which satisfy the conditions (47). To do this an infinite set of constants may be introduced by putting

$$g_n(\xi) = \sum_{p=1}^{\infty} B_p\, g_n^{(p)}(\xi) \quad (n = 1, 2, 3, \ldots) \tag{52}$$

in (38). Then the aggregate solution

$$\{g_n^{(p)}(\xi)\} \quad (n = 1, 2, 3, \ldots) \tag{53}$$

satisfies (38) and represents, for fixed p, one of a set of linearly independent solutions obtained by assigning the values $p = 1, 2, 3, \ldots$. The linear independence is ensured by assigning independent boundary conditions in each mode of solution.

To consider in detail how a typical solution is obtained, let $\xi = \xi_1$ be the sufficiently large value of ξ referred to in the last section. For $\xi > \xi_1$ each $g_n^{(p)}$ is given by (49) with the constants D_n chosen to make them satisfy suitable imposed conditions at $\xi = \xi_1$, e.g. for fixed p,

$$g_p^{(p)}(\xi_1) = 1, \quad g_n^{(p)}(\xi_1) = -1 \quad (n \neq p;\, n = 1, 2, 3, \ldots). \tag{54}$$

These are the independent boundary conditions referred to above; they are distinct as p varies and therefore define a distinct numerical mode of solution for each value of p, valid for $\xi \geqslant \xi_1$.

Each of these modes of solution is now completed by solving equations (38) for $\xi < \xi_1$ by step-by-step integration, working backwards from $\xi = \xi_1$ to $\xi = 0$, making the inner functions and their first derivatives continuous with the outer solution at $\xi = \xi_1$. Denoting a typical equation of the set by

$$g'' + K(\xi) = 0,$$

a suitable difference formula, using a step h, is

$$g(\xi-2h)+g(\xi+2h)-2g(\xi)+\tfrac{4}{3}h^2\{K(\xi-h)+K(\xi)+K(\xi+h)\} = 0,$$

for it allows all equations of the set to be integrated simultaneously, starting the integration by calculating $g(\xi_1-h)$ from appropriate values at ξ_1, ξ_1+h, ξ_1+2h and ξ_1+3h. The truncation error in the formula is $O(h^6)$.

From the computed sets of fundamental solutions (53), the third and final stage of the iteration is to determine the constants B_n. Substituting from (52) into (24) and thence into (47), the B_n must satisfy the infinite set of linear equations

$$\left.\begin{aligned}\sum_{p=1}^{\infty}\Phi_{n,p}B_p &= -1 \quad (n=1) \qquad (55a)\\ &= 0 \quad (n=2,3,4,\ldots),\qquad (55b)\end{aligned}\right\}$$

where
$$\Phi_{n,p} = \frac{1}{2}\int_0^{\infty}\Psi_{n,p}\,e^{-n\xi}\,d\xi, \tag{56}$$

with

$$\Psi_{n,p} = \sum_{r=1}^{\infty} g_r^{(p)}\{\mathscr{I}_{n-r}-\mathscr{I}_{n+r})\cosh 2\xi - \tfrac{1}{2}(\mathscr{I}_{n-r-2}+\mathscr{I}_{n-r+2}-\mathscr{I}_{n+r-2}-\mathscr{I}_{n+r+2})\}. \tag{57}$$

Values of the integral $\mathscr{I}_n(\xi)$ are obtained using numerical integration, and further numerical integration determines $\Phi_{n,p}$.

Equations (55*b*) are homogeneous in the constants B_n and serve to determine the ratios

$$B_n/B_1 = B'_n \quad (n=2,3,4,\ldots). \tag{58}$$

Having found them, we may determine B_1 from (55*a*), although in practice a slight modification of this last step is made. Using the values of $B_n\,(n \neq 1)$ we can re-write (52) as

$$g_n(\xi) = B_1 G_n(\xi) \quad (n=1,2,3,\ldots), \tag{59}$$

with
$$G_n(\xi) = g_n^{(1)}(\xi) + \sum_{p=2}^{\infty} B'_p g_n^{(p)}(\xi).$$

The constant B_1 is now found by substituting (59) into (24) with $n = 1$ and then substituting $r_1(\xi)$ into the first of (47). It should be noted that the final solution $\{G_n(\xi)\}$ can be made arbitrary to the extent of any single scaling factor, this merely altering the definition of B_1. It is convenient to choose this factor so that $G_1(0) = 1$; the drag coefficient then becomes $C_D = -\pi B_1/R$.

The functions $r_n(\xi)$ are now computed by substituting from (59) into (24). In practice the functions $R_n(\xi)$ defined by

$$r_n(\xi) = -B_1 R_n(\xi) \quad (n=1,2,3,\ldots) \tag{60}$$

are recorded. At this stage a cycle of the iterative process is complete and the next cycle is entered by re-computing the functions f_n.

In practice the upper limit in the integral (56) is replaced by a finite number $\xi = \delta$, beyond which the integrand is negligible. During the course of calculations it soon becomes apparent that δ decreases as the Reynolds number R increases. This introduces the notion of a boundary-layer thickness into the calculations and it can, in fact, be shown that for large enough R, $\delta \propto R^{-\frac{1}{2}}$. Briefly,† this may be done by introducing the changes of variable $\xi = \delta z, f_n = B_1 \delta^2 F_n$ in the equations (21) and (38), when they become respectively

$$d^2F_n/dz^2 - \delta^2 n^2 F_n = R_n(\delta z), \tag{61}$$

$$\frac{d^2 g_n}{dz^2} - \delta^2 n^2 g_n + \delta^2 \sum_{p=1}^{\infty} k_{n,p}\, g_p = 0. \tag{62}$$

If it is assumed that $\delta \to 0$ as $R \to \infty$ and, further, that the right-hand side of (61) becomes a function of z alone, solutions of (61) may be found which become functions of z alone as $R \to \infty$. Substituting these functions and their derived functions into (44) and (45) and considering the form they take, it is fairly easy to deduce that for large enough R

$$\delta^2 k_{n,p} \sim RB_1 \delta^3 \lambda_{n,p}(z) + R^2 B_1^2 \delta^6 \mu_{n,p}(z),$$

and this can be made a function of z alone, $K_{n,p}(z)$, by choosing

$$RB_1 \delta^3 = c', \tag{63}$$

where c' is a definite numerical constant. Solutions of (62) can now be found which are functions of z alone, the term in n^2 in (62) tending to zero with δ.

Finally, it may then be deduced that the functions $\Psi_{n,p}$ are of order unity and, replacing the upper limit in (56) by δ and changing to the variable z, $\Phi_{n,p}$ is of order δ. Thus from (55) the constants B_n are of order δ^{-1} and, in particular,

$$B_1 = a\delta^{-1}, \tag{64}$$

where a is a constant. Combining this with (63) shows that $\delta = O(R^{-\frac{1}{2}})$. Since $\Psi_{n,p}$ is of order unity the functions R_n are functions of z alone, and this completes the process.

A solution of the limiting form of (61) and (62), i.e. with the terms in n^2 suppressed, has been obtained. Here the functions $K_{n,p}(z)$ are calculated from the same formula (43) but with the definitions (42) modified. The modifications are that β_n now defines a function of z by the same formula (41), that f_n' stands for differentiation with regard to z, that the terms involving β_n in the definitions of b_n and c_n are suppressed and finally, that R be replaced by the numerical constant c' defined by (63). The constant c' can be chosen at will; this merely alters the boundary-layer thickness in the co-ordinate z. Finally, it follows from (63) and (64) that $B_1 = a(a/c')^{\frac{1}{2}} R^{\frac{1}{2}}$ and hence $C_D = -\pi a(a/c')^{\frac{1}{2}} R^{-\frac{1}{2}}$.

4. Calculated results

The results given in the present section were computed on a Ferranti Mercury Computer using the numerical methods previously described. In practical calculations, the series (18) and (20) have to be truncated, i.e. only a finite number

† This is considered in more detail by Dennis & Dunwoody (1964).

of $f_n(\xi)$ and $g_n(\xi)$ may be considered. In the present case limitations were imposed by the available computer storage space. The storage problem is not serious for the $f_n(\xi)$, but if the calculations extend to m terms of the $g_n(\xi)$ one has currently to store m^2 component numerical solutions [viz. each $g_n^{(p)}(\xi)$ for $n, p = 1, 2, \ldots, m$] together with m^2 functions $k_{n,p}(\xi)$. In fact the calculations had to be limited to $m = 5$; the effect on the results of this truncation is discussed in the final section.

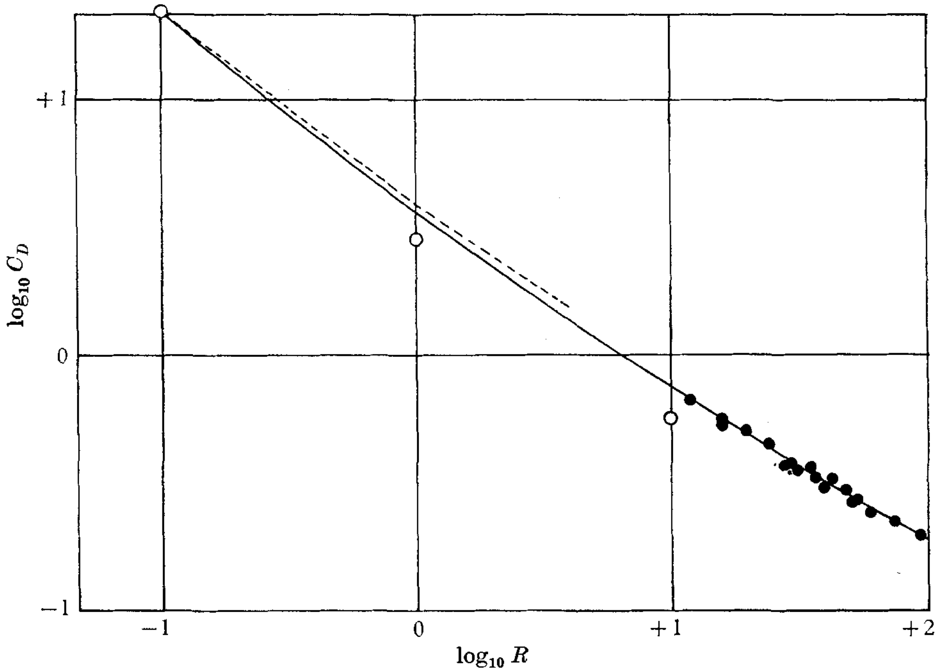

FIGURE 1. Comparison of drag coefficients for the range $R = 0{\cdot}1$ to $R = 100$. ——, Dennis & Dunwoody; – – – –, Oseen theory (Tomotika & Aoi); ○, Janssen; ●, experimental (Janour).

R	C_D	R	C_D
10	0·748	500	0·0731
15	0·581	1000	0·0502
20	0·483	2000	0·0341
40	0·316	5000	0·0206
100	0·188	10000	0·0141

TABLE 1. Frictional drag coefficient

The iterative process for obtaining numerical solutions was always started by assuming

$$r_1(\xi) = -1, \quad r_n(\xi) = 0 \quad (n = 2, 3, 4, \ldots)$$

in (21). The initial assumption for the $f_n(\xi)$ is therefore

$$f_1(\xi) = \cosh\xi - 1, \quad f_n(\xi) = 0 \quad (n \neq 1).$$

This satisfies the inner boundary conditions for ψ and also (to some extent) the outer conditions since, as $\xi \to \infty$, it leads to $\psi(\xi, \eta) \sim \frac{1}{2} e^{\xi} \sin\eta$. On the whole, even at higher Reynolds numbers, less than ten iterations were needed to achieve convergence to the final solution.

Calculated values of the frictional drag coefficient C_D over a wide range of Reynolds numbers are given in table 1. In figure 1 the results for the range $R = 0{\cdot}1$ to 100 are compared with Oseen theory at the lower end, and with Janour's experimental results for $R > 12$. The approach to Oseen theory for very low values of R may be noticed, and a more precise numerical comparison is given in table 2. The Oseen values have been calculated from the formula

$$C_D = \frac{4\pi}{RS}\left[1-\frac{1}{S}(S^2-S-\tfrac{5}{12})\frac{R^2}{128}-\frac{1}{S^2}(S^4+\tfrac{1}{12}S^3-\tfrac{23}{24}S^2-\tfrac{133}{360}S-\tfrac{25}{144})\frac{R^4}{(128)^2}\right]$$

given by Tomotika & Aoi. Here

$$S = 1-\gamma-\log(R/16),$$

where γ is Euler's constant. The difference ranges from $\frac{1}{2}\%$ at $R = 0{\cdot}1$ to 14% at $R = 4$, increasing in a fairly uniform way.

R	Calculated	Oseen	Janssen
0·1	22·66	22·85	22·23
0·2	12·80	13·06	—
0·4	7·33	7·61	—
0·6	5·34	5·61	—
1·0	3·64	3·87	2·80
2·0	2·20	2·39	—
4·0	1·36	1·55	—

TABLE 2. Comparison of C_D for low R

By contrast, the comparison of present results with Janssen's, obtained by electrical analogue methods, is less uniform. His values at $R = 1$ and at $R = 10$ ($C_D = 0{\cdot}570$) seem low (cf. the experimental trend at $R = 10$). It is possible that the length between lattice points ($= \frac{1}{8}\pi$) used by Janssen is too high at the higher values of R. It was found necessary in the present method to use the respective steps $h = 0{\cdot}15$, $0{\cdot}1$ and $0{\cdot}05$ in ξ at the three values $R = 0{\cdot}1$, 1 and 10. Moreover, as previously noted, Janssen's satisfaction of the outer boundary conditions does not appear to be correct.

Figure 2 shows a comparison of calculated values of C_D for $R > 100$ with Janour's experimental results. The theoretical result

$$C_D \approx \frac{1{\cdot}328}{R^{\frac{1}{2}}}+\frac{4{\cdot}12}{R}$$

given by Kuo is also shown.

Van Dyke (1962) suggests that the coefficient 4·12 in this result should be approximately 5·3; on this basis the agreement with present results would be better, e.g. at $R = 100$, Kuo's result gives $C_D = 0{\cdot}174$, while the corrected result gives $C_D = 0{\cdot}186$, compared with the present value of 0·188.†

† This was pointed out by a referee.

A result for large R based on the method indicated in the last section has been obtained. Taking $c' = -42{\cdot}93$ (note that B_1 in (63) is negative) a value $a = -2{\cdot}031$ resulted, leading to

$$\delta = 4{\cdot}60R^{-\frac{1}{2}}, \quad B_1 = -0{\cdot}422R^{\frac{1}{2}}, \quad C_D = 1{\cdot}389R^{-\frac{1}{2}}.$$

The limiting value of the frictional drag coefficient is therefore a little higher than the value obtained from the Blasius theory.

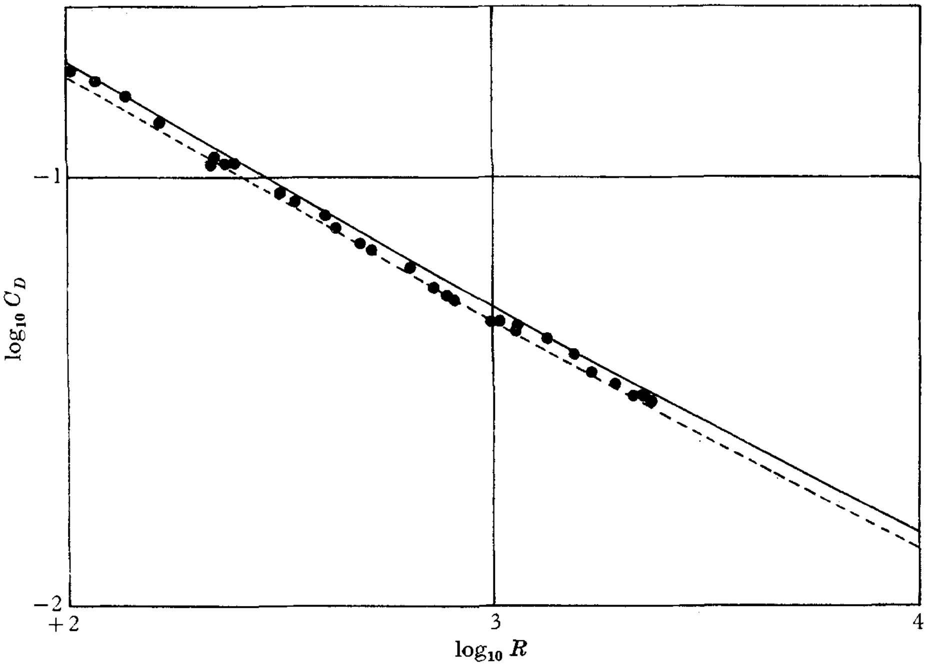

FIGURE 2. Comparison of drag coefficients for the range $R = 100$ to $R = 10{,}000$. ——, Dennis & Dunwoody; – – – –, Kuo (theoretical); ●, experimental (Janour).

The local variation of skin friction over the plate is obtained from equation (51). The best way to obtain ζ is to compare the governing equation (12) with its transformed analogue (21), whence it follows that

$$\zeta(\xi, \eta) = \frac{2}{\cosh 2\xi - \cos 2\eta} \sum_{n=1}^{\infty} r_n(\xi) \sin n\eta. \tag{65}$$

Putting $\xi = 0$ gives ζ_0. The theoretical form of the variation of skin friction near the edges of the plate can be found from (65). Near the leading edge ($\xi = 0$, $\eta = \pi$) we may put $\eta = \pi - \eta'$ in (65) and expand in powers of η', which gives

$$\zeta_0(\eta') = A/\eta' + O(\eta'),$$

where
$$A = \sum_{n=1}^{\infty} (-1)^{n+1} n r_n(0).$$

On the plate we also have

$$X = c(1 - \cos\eta') \sim \tfrac{1}{2}c\eta'^2$$

for small η', where X is measured along the plate from the leading edge. Hence, near $X = 0$

$$c_f \approx \frac{A}{R}\left(\frac{2c}{X}\right)^{\frac{1}{2}}. \tag{66}$$

In particular as $R \to \infty$, $A = O(R^{\frac{1}{2}})$ from the results of the previous section and this result becomes

$$c_f \approx A'(\nu/UX)^{\frac{1}{2}}, \tag{67}$$

where $A'\,(= AR^{-\frac{1}{2}})$ is a numerical constant. This is in agreement with the Blasius theory. A similar result holds near the trailing edge. Here we have only to replace X by $2c - X$ and A by the quantity

$$B = \sum_{n=1}^{\infty} nr_n(0)$$

in (66). As $R \to \infty$, the corresponding form of (67) is then

$$c_f \approx B'\{\nu/U(2c-X)\}^{\frac{1}{2}}, \tag{68}$$

where $B = B'R^{\frac{1}{2}}$.

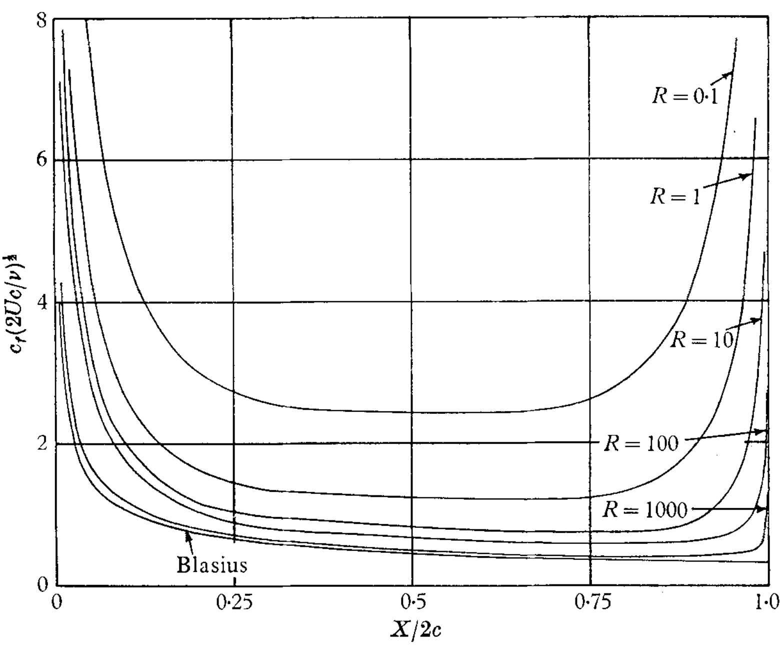

FIGURE 3. Local distribution of skin friction over the plate. The Blasius curve is $c_f = 0{\cdot}332(\nu/UX)^{\frac{1}{2}}$.

The constants A and B cannot be found with any great accuracy from the present solution because of the limited number of terms computed and the rather severe requirement of calculating the summations involving the terms $nr_n(0)$. Qualitatively, however, the behaviour of A' and B' with increasing R is quite striking, viz. B' decreases and is $\ll A'$ for large R. Except very near the end-points, the distribution of skin friction over the plate can be found with reasonable accuracy from the series (65). Some results are given in figure 3. The approach to the Blasius solution as R becomes large is clearly seen, particularly the diminishing effect of the trailing edge singularity. This latter tendency has already been noted by Janssen over the range $R = 0{\cdot}1$ to 10.

The variation of boundary-layer thickness near the leading edge of the plate is also in agreement with the Blasius theory. In general the thickness $\xi = \delta$ corresponds to a thickness $y = \delta' \sim c\delta \sin \eta'$. Using the results of the calculations for large R we have, near the leading edge,

$$\delta' \approx c\delta(2X/c)^{\frac{1}{2}} = 4{\cdot}60(\nu X/U)^{\frac{1}{2}}.$$

It may be noted that the present manner of choosing the thickness δ, especially suitable in the present approach, gives an estimate of δ' within the numerical range of the more conventional estimates (cf. in particular the numerical coefficient 5·0 given by Schlichting 1960).

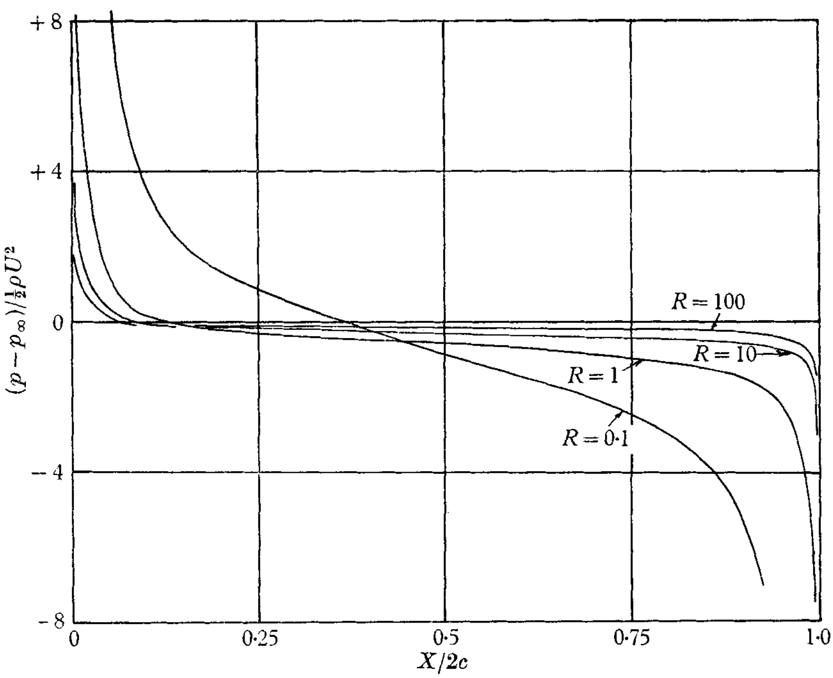

FIGURE 4. Pressure distribution over the plate.

The pressure field may be obtained by integrating the equations of motion in the (ξ, η)-plane. First, the pressure gradient in the ξ-direction is integrated along the path $\eta = \frac{1}{2}\pi$ from $\xi = 0$ to $\xi = \infty$. This gives the pressure p_0 at the centre of the plate, and the pressure at any point $\xi = 0, \eta = \eta_1$ on the plate is then obtained by integrating the pressure gradient in the η-direction along $\xi = 0$, which yields

$$p - p_0 = \frac{U\rho\nu}{c} \int_{\frac{1}{2}\pi}^{\eta_1} \left(\frac{\partial\zeta}{\partial\xi}\right)_0 d\eta.$$

Some practical difficulties were encountered in calculating the pressure distribution on the plate. The reason is that whereas the series on the right-hand side of (65) converges well enough to obtain reasonable values of ζ, the series obtained by differentiation with regard to ξ is less satisfactory. In the end it was found necessary to make use of finite-difference equations (in both the ξ and η directions) on and near the plate in order to obtain smooth values of $(\partial\zeta/\partial\xi)_0$. Some calculated pressure distributions over the plate are shown in figure 4. Although we do not claim high accuracy for these, the results indicate clearly the tendency of pressure gradients to become zero (except at the edges of the plate) as R increases. This agrees with Blasius theory.

5. Discussion

The mathematical assumptions in the present paper are not greatly dissimilar from those made in the standard theoretical treatment of Oseen's linearized equations. The essentially new point is the satisfaction of the full governing equation (11) for the scalar vorticity ζ by means of equations (15) and (18); and also the fact that all solutions of the relevant equations are carried out numerically. In this connexion it has to be assumed that some theoretical meaning can be attached to the system of differential equations (38) and their numerical solutions. This assumption seems reasonable, especially in view of the fact that the computed solutions give results which are consistent with Oseen theory as $R \to 0$ and with the Blasius theory as $R \to \infty$.

The edges of the plate are clearly points of non-uniformity of convergence of the series (18). The convergence of this series does not, however, have any real bearing on the numerical process of solution. It is the fundamental solutions of (38) [viz. the $g_n^{(p)}(\xi)$] which are actually computed and these are subsequently combined in the form (18) by the satisfaction of the integral condition (47). In effect this is a condition on the $r_n(\xi)$, not the $g_n(\xi)$. Thus the important properties of the function ζ (e.g. the leading and trailing edge singularities) are deduced from (65); and the series (18) plays no real part in the solution. In this sense the method of analysis is very similar to the method of separation of variables.

The numerical model used in the present paper is, of course, a restricted one. However, although the basic calculations were restricted to only five terms in the series (18) it is not difficult, once this basic solution is computed, to obtain estimates of further terms in (18) using a more simplified calculation process requiring little extra computer storage space. Such estimates indicate that the results presented here are reasonably accurate. In particular, the average skin friction on the plate depends only on the constant B_1 and this does not seem to be very sensitive to the number of terms taken into account in the series (18). The principal source of error in the present method arises, in fact, from the difficulty of making the join of the inner (finite-difference) solution with the outer solution of equations (38) at a large enough value of ξ_1, since the inner solutions lose accuracy if ξ_1 is too large.

The only serious disagreement with previous results is the discrepancy with Janssen's drag values at $R = 1$ and 10. As a check on our own calculations, solutions at $R = 20$ and 40 are at present being computed using two-dimensional relaxation methods.

An account of some of the results of the present paper forms part of a Ph.D. thesis submitted by one of us (J. D.) to Queen's University, Belfast. The range of Reynolds numbers has been extended in the present paper and some of the previous results have been recalculated using improved numerical methods (viz. smaller integration steps, allowing a better join of inner and outer solutions at $\xi = \xi_1$).

Some of the work described above has been carried out as part of the research programme of the National Physical Laboratory, and is published by permission of the Director of the Laboratory.

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